Asymptotic behaviors for distribution dependent SDEs driven by fractional Brownian motions

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Outline

1. The problem, and related works
2. Main Results
   - Large deviation principle (LDP)
   - Moderate deviation principle (MDP)
   - Central limit theorem (CLT)
3. Proofs

The problem, and related works

Distribution dependent stochastic differential equations (SDEs), also called McKean-Vlasov or mean-field SDEs, is of the form:

$$\mathrm{d}X_t = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W_t, \quad X_0 = \xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P}).$$

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- W. Liu, Y. Song, J. Zhai and T. Zhang, *Large and moderate deviation principles for McKean-Vlasov SDEs with jumps*, PA, 2022.
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- A. Budhiraja and X. Song, *Large deviation principles for stochastic dynamical systems with a fractional Brownian noise*, arXiv:2006.07683.
- X. Gu and Y. Song, *Large and moderate deviation principles for path-distribution-dependent stochastic differential equations*, Discrete Contin. Dyn. Syst. Ser. S, 2023.
- K. Liu, W. Liu, H. Qiao and F. Zhu, *Asymptotic behaviors of small perturbation for multivalued McKean-Vlasov stochastic differential equations*, 2023.

Our concerned equation:

$$\mathrm{d}X_t = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, \mathscr{L}_{X_t})\mathrm{d}B_t^H, \quad X_0 = x, \tag{1}$$

where $\mathscr{L}_{X_t^\epsilon}$ denotes the law of X_t^ϵ, $\epsilon > 0$ is a small parameter, B^H is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, the coefficients b and σ fulfill some appropriate conditions given in later sections. Moreover, the integral with respect to B^H is interpreted in the Wiener sense due to the fact that $\sigma(\cdot, \mathscr{L}_{X_\cdot^\epsilon})$ is deterministic.
The main purpose of this paper is to study the LDP, MDP and CLT of (1) when $\epsilon \to 0$. More precisely, let X^0 be the limit of X^ϵ in some sense, we are going to investigate the asymptotic behaviors for the path of the form

$$Y_t^\epsilon := \frac{X_t^\epsilon - X_t^0}{\epsilon^H \kappa(\epsilon)}, \quad t \in [0, T].$$

• In the case of the LDP, namely $\kappa(\epsilon) = 1/\epsilon^H$, we show that X^ϵ satisfies the LDP with speed ϵ^{2H}.
• In the case of the CLT, namely $\kappa(\epsilon) = 1$, we prove that as $\epsilon \to 0$, $\frac{X^\epsilon - X^0}{\epsilon^H}$ converges to a stochastic process which solves a linear equation involving the Lions derivative of the coefficient b.
• In the case of the MDP, namely $\kappa(\epsilon) \to \infty$ and $\epsilon^H \kappa(\epsilon) \to 0$ as $\epsilon \to 0$, we derive that Y^ϵ satisfies LDP with speed $\kappa^{-2}(\epsilon)$.
Here, let us point out that the MDP for X^ϵ refers to the LDP for Y^ϵ since the scaling by $\epsilon^H \kappa(\epsilon)$ means that the MDP is in the regime between the LDP and the CLT.

For some fixed $H \in (1/2, 1)$. we consider $(\Omega, \mathscr{F}, \mathbb{P})$ the canonical probability space associated with fractional Brownian motion with Hurst parameter H such that the canonical process $\{B_t^H; t \in [0,T]\}$ is a d-dimensional fractional Brownian motion with Hurst parameter H. Recall that $B^H = (B^{H,1}, \cdots, B^{H,d})$ is a centered Gaussian process, whose covariance structure is defined by

$$\mathbb{E}\left(B_t^{H,i} B_s^{H,j}\right) = R_H(t,s)\delta_{i,j}, \quad s,t \in [0,T], \; i,j = 1, \cdots, d$$

with $R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.
We denote by $\mathscr{E}$ the set of step functions on $[0,T]$ with values in $\mathbb{R}^d$. Let $\mathcal{H}$ be the Hilbert space defined as the completion of $\mathscr{E}$ with respect to the scalar product

$$\left\langle (\mathbb{I}_{[0,t_1]}, \cdots, \mathbb{I}_{[0,t_d]}), (\mathbb{I}_{[0,s_1]}, \cdots, \mathbb{I}_{[0,s_d]}) \right\rangle_{\mathcal{H}} = \sum_{i=1}^{d} R_H(t_i, s_i).$$

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,r) K_H(s,r) \mathrm{d}r,$$

where $K_H(t,s)$ is the square integrable kernel given by

$$K_H(t,s) = C_H s^{\frac{1}{2}-H} \int_s^t (r-s)^{H-\frac{3}{2}} r^{H-\frac{1}{2}} \mathrm{d}r, \quad t > s$$

with $C_H = \sqrt{\frac{H(2H-1)}{\mathcal{B}(2-2H, H-1/2)}}$ and $\mathcal{B}$ standing for the Beta function.

Let $(e_1, \cdots, e_d)$ designate the canonical basis of $\mathbb{R}^d$, one can introduce the linear operator $K_H^* : \mathscr{E} \to L^2([0,T], \mathbb{R}^d)$ defined by

$$K_H^*(\mathrm{I}_{[0,t]} e_i) = K_H(t, \cdot) e_i.$$

$\langle K_H^* \psi, K_H^* \phi \rangle_{L^2([0,T],\mathbb{R}^d)} = \langle \psi, \phi \rangle_{\mathcal{H}}$ holds for all $\psi, \phi \in \mathscr{E}$. There exists a d-dimensional Wiener process W defined on $(\Omega, \mathscr{F}, \mathbb{P})$ such that B^H has the following Volterra-type representation

$$B_t^H = \int_0^t K_H(t,s) \mathrm{d}W_s, \quad t \in [0,T]. \tag{2}$$

Moreover, K_H^* has the following representations: for any $\psi, \phi \in \mathcal{H}$,

$$(K_H^* \psi)(t) = \int_t^T \psi(s) \frac{\partial K_H(s,t)}{\partial s} \mathrm{d}s$$

and

$$\langle K_H^* \psi, K_H^* \phi \rangle_{L^2([0,T],\mathbb{R}^d)} = \langle \psi, \phi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T |t-s|^{2H-2} \langle \psi(s), \phi(t) \rangle_{\mathbb{R}^d} \mathrm{d}s \mathrm{d}t. \tag{3}$$

As a consequence, for any $\psi \in L^2([0,T], \mathbb{R}^d)$, one has

$$\|\psi\|_{\mathcal{H}}^2 \le 2HT^{2H-1} \|\psi\|_{L^2}^2. \tag{4}$$

Besides, one can show that $L^{1/H}([0,T], \mathbb{R}^d) \subset \mathcal{H}$.

Next, we define the operator $K_H : L^2([0,T],\mathbb{R}^d) \to I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d))$ by

$$(K_H f)(t) = \int_0^t K_H(t,s)f(s)\mathrm{d}s,$$

where I_{0+}^{α} is the left-sided fractional Riemann-Liouville integral operator of order $\alpha(>0)$ given by

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y, \ \ f \in L^1([0,T],\mathbb{R}^d), \ x \in (0,T). \tag{5}$$

Let us mention that the space $I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d))$ is the fractional version of the Cameron-Martin space. Finally, we denote by $R_H = K_H \circ K_H^* : \mathcal{H} \to I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d))$ the operator

$$(R_H \psi)(t) = \int_0^t K_H(t,s)(K_H^* \psi)(s)\mathrm{d}s. \tag{6}$$

Since $I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d)) \subset C^H([0,T],\mathbb{R}^d)$, we know that for any $\psi \in \mathcal{H}$, $R_H\psi$ is Hölder continuous of order H, i.e.

$$R_H \psi \in C^H([0,T],\mathbb{R}^d), \ \ \psi \in \mathcal{H}. \tag{7}$$

$$(R_H \psi)(t) = \int_0^t \left(\int_0^s \frac{\partial K_H}{\partial s}(s,r)(K_H^* \psi)(r)\mathrm{d}r \right) \mathrm{d}s. \tag{8}$$

The Lions derivative

For any $\theta \in [1, \infty)$, $\mathscr{P}_\theta(\mathbb{R}^d)$ stands for the set of θ-integrable probability measures on $\mathbb{R}^d$, and define the L^θ-Wasserstein distance on $\mathscr{P}_\theta(\mathbb{R}^d)$ as follows

$$\mathbb{W}_\theta(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\theta \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{\theta}}, \quad \mu, \nu \in \mathscr{P}_\theta(\mathbb{R}^d).$$

Here $\mathscr{C}(\mu, \nu)$ denotes the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν.

Definition

Let $f : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $g : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

(1) f is called L-differentiable at $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, if the functional

$$L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto f(\mu \circ (\mathrm{Id} + \phi)^{-1}))$$

is Fréchet differentiable at $0 \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$. That is, there exists a unique $\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that

$$\lim_{\|\phi\|_{L^2_\mu} \to 0} \frac{f(\mu \circ (\mathrm{Id} + \phi)^{-1}) - f(\mu) - \langle \gamma, \phi \rangle_\mu}{\|\phi\|_{L^2_\mu}} = 0,$$

where $\langle \gamma, \phi \rangle_\mu = \int_{\mathbb{R}^d} \langle \gamma(x), \phi(x) \rangle \mu(\mathrm{d}x)$. In this case, γ is called the L-derivative of f at μ and denoted by $D^L f(\mu)$.

Definition

(2) f is called L-differentiable on $\mathscr{P}_2(\mathbb{R}^d)$, if the L-derivative $D^L f(\mu)$ exists for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Furthermore, if for every $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a μ-version $D^L f(\mu)(\cdot)$ such that $D^L f(\mu)(x)$ is jointly continuous in $(\mu, x) \in \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$, we denote $f \in C^{(1,0)}(\mathscr{P}_2(\mathbb{R}^d))$.

(3) g is called differentiable on $\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, if for any $(x, \mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, $g(\cdot, \mu)$ is differentiable and $g(x, \cdot)$ is L-differentiable. Furthermore, if $\nabla g(\cdot, \mu)(x)$ and $D^L g(x, \cdot)(\mu)(y)$ are jointly continuous in $(x, y, \mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, we denote $g \in C^{1,(1,0)}(\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$.

Well-posedness of DDSDE

DDSDE driven by fractional Brownian motion of the form:

$$\mathrm{d}X_t = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, \mathscr{L}_{X_t})\mathrm{d}B_t^H, \quad X_0 = x, \tag{9}$$

where the coefficients $b : [0,T] \times \mathbb{R}^d \times \mathscr{P}_\theta(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : [0,T] \times \mathscr{P}_\theta(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ with $\theta \in [1,2]$.

(H1) There exists a non-decreasing function $K(t)$ such that for any $t \in [0,T], x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_\theta(\mathbb{R}^d)$,

$$|b(t,x,\mu) - b(t,y,\nu)| \leq K(t)(|x-y| + \mathbb{W}_\theta(\mu,\nu)), \quad \|\sigma(t,\mu) - \sigma(t,\nu)\| \leq K(t)\mathbb{W}_\theta(\mu,\nu),$$

and

$$|b(t,0,\delta_0)| + \|\sigma(t,\delta_0)\| \leq K(t).$$

For any $p \geq 1$, let $\mathcal{S}^p([0,T])$ be the space of $\mathbb{R}^d$-valued, continuous $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes ψ on $[0,T]$ satisfying

$$\|\psi\|_{\mathcal{S}^p} := \left(\mathbb{E} \sup_{t\in[0,T]} |\psi_t|^p\right)^{1/p} < \infty,$$

Well-posedness of DDSDE

Definition

A stochastic process $X = (X_t)_{0\le t\le T}$ on $\mathbb{R}^d$ is called a solution of (9), if $X \in \mathcal{S}^p([0,T])$ and $\mathbb{P}$-a.s.,

$$X_t = \xi + \int_0^t b(s, X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma(s, \mathscr{L}_{X_s})\mathrm{d}B_s^H, \quad t \in [0,T].$$

Theorem (Fan-Huang-Suo-Yuan, SPA, 2022)

Suppose that $\xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$ with $p \ge \theta$ and one of the following conditions:

(I) $H \in (1/2, 1)$, b, σ satisfy **(H1)** and $p > 1/H$;

(II) $H \in (0, 1/2)$, b satisfies **(H1)** and $\sigma(t,\mu)$ does not depend on (t,μ).

Then equation (9) has a unique solution $X \in \mathcal{S}^p([0,T])$.

- L. Galeati, F.A. Harang and A. Mayorcas, *Distribution dependent SDEs driven by additive fractional Brownian motion*, PTRF, 2022.

Reference equation

$C_b(\mathscr{E})$ denotes the set of all bounded continuous functions $f : \mathscr{E} \to \mathbb{R}$ with the norm $\|f\|_\infty := \sup_{x \in \mathscr{E}} |f(x)|$, where $\mathscr{E}$ is a Polish space with the Borel σ-field $\mathcal{B}(\mathscr{E})$. Let

$$\mathcal{A} = \left\{ \phi : \phi \text{ is } \mathbb{R}^d\text{-valued } \mathcal{F}_t\text{-predictable process and } \|\phi\|_{\mathcal{H}}^2 < \infty \ \mathbb{P}\text{-a.s.} \right\},$$

and for each $M > 0$, let

$$S_M = \left\{ h \in \mathcal{H} : \frac{1}{2}\|h\|_{\mathcal{H}}^2 \le M \right\}.$$

It is obvious that S_M endowed with the weak topology is a Polish space. Besides, define

$$\mathcal{A}_M := \{\phi \in \mathcal{A} : \phi(\omega) \in S_M, \ \mathbb{P}\text{-a.s.}\}.$$

For any fixed $\mu_\cdot \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$, we introduce the following reference equation:

$$\mathrm{d}\tilde{X}_t = b(t,\tilde{X}_t,\mu_t)\mathrm{d}t + \sigma(t,\mu_t)\mathrm{d}B_t^H, \quad 0 \le t \le T \tag{10}$$

with initial value $\tilde{X}_0 = y \in \mathbb{R}^d$.

Lemma (Lemma 1)

Suppose that **(H1)** *holds. Then for any* $\mu_\cdot \in C([0,T];\mathscr{P}_2(\mathbb{R}^d))$, *there is a measurable map* $\mathcal{G}_\mu : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ *such that*

$$\tilde{X}_\cdot = \mathcal{G}_\mu(B_\cdot^H).$$

Moreover, for each $h \in \mathcal{A}_M$, *define*

$$\tilde{X}_\cdot^h := \mathcal{G}_\mu\left(B_\cdot^H + (R_H h)(\cdot)\right),$$

then $\tilde{X}^h$ *satisfies the following equation*

$$\begin{aligned}\tilde{X}_t^h =& y + \int_0^t b(s,\tilde{X}_s^h,\mu_s)\mathrm{d}s + \int_0^t \sigma(s,\mu_s)\mathrm{d}(R_H h)(s) \\ &+ \int_0^t \sigma(s,\mu_s)\mathrm{d}B_s^H, \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s..\end{aligned} \tag{11}$$

We now consider the following distribution dependent SDE:

$$\mathrm{d}X_t = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, \mathscr{L}_{X_t})\mathrm{d}B_t^H, \quad X_0 = x \in \mathbb{R}^d, \quad t \in [0, T]. \tag{12}$$

From the above lemma, it easily follows the following result.

Lemma (Lemma 2)

Suppose that $y = x$ *and* $\mu_t = \mathscr{L}_{X_t}, t \in [0, T]$ *for equation* (10) *and* **(H1)** *holds. Then the solution* X *of equation* (12) *satisfies* $X_\cdot = \mathcal{G}_{\mathscr{L}_X}(B_\cdot^H)$, *where* $\mathcal{G}_{\mathscr{L}_X}$ *is given in Lemma 1 with* $\mu = \mathscr{L}_X$. *Moreover, for any* $h \in \mathcal{A}_M$, *let*

$$X_\cdot^h = \mathcal{G}_{\mathscr{L}_X}\left(B_\cdot^H + (R_H h)(\cdot)\right),$$

then X^h *satisfies the following equation*

$$\begin{aligned} X_t^h = & x + \int_0^t b(s, X_s^h, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma(s, \mathscr{L}_{X_s})\mathrm{d}(R_H h)(s) \\ & + \int_0^t \sigma(s, \mathscr{L}_{X_s})\mathrm{d}B_s^H, \quad t \in [0, T], \quad \mathbb{P}\text{-}a.s.. \end{aligned}$$

In this talk, our main objective is to study asymptotic behaviors for following DDSDEs driven by fractional Brownian motions. For any $\epsilon > 0$,

$$\mathrm{d}X_t^\epsilon = b(t, X_t^\epsilon, \mathscr{L}_{X_t^\epsilon})\mathrm{d}t + \epsilon^H \sigma(t, \mathscr{L}_{X_t^\epsilon})\mathrm{d}B_t^H, \quad X_0^\epsilon = x. \tag{13}$$

According to Lemma 2, there exists a measurable map $\mathcal{G}^\epsilon := \mathcal{G}_{\mathscr{L}_{X^\epsilon}}$ such that $X_\cdot^\epsilon = \mathcal{G}^\epsilon(\epsilon^H B_\cdot^H)$. Furthermore, for every $h^\epsilon \in \mathcal{A}_M$, let

$$X_\cdot^{\epsilon,h^\epsilon} := \mathcal{G}^\epsilon\left(\epsilon^H B_\cdot^H + (R_H h^\epsilon)(\cdot)\right), \tag{14}$$

then X^{ϵ,h^ϵ} satisfies the following equation

$$\begin{aligned} X_t^{\epsilon,h^\epsilon} =& x + \int_0^t b(s, X_s^{\epsilon,h^\epsilon}, \mathscr{L}_{X_s^\epsilon})\mathrm{d}s + \int_0^t \sigma(s, \mathscr{L}_{X_s^\epsilon})\mathrm{d}(R_H h^\epsilon)(s) \\ &+ \epsilon^H \int_0^t \sigma(s, \mathscr{L}_{X_s^\epsilon})\mathrm{d}B_s^H, \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s.. \end{aligned} \tag{15}$$

Besides, we need the following result.

Proposition

Suppose that **(H1)** *holds. Then there exists a unique function* $\{X_t^0\}_{t\in[0,T]}$ *such that*

(i) $X^0 \in C([0,T];\mathbb{R}^d)$,

(ii) X^0 *satisfies the following deterministic equation*

$$X_t^0 = x + \int_0^t b(s, X_s^0, \mathscr{L}_{X_s^0})\mathrm{d}s, \quad t \in [0,T]. \tag{16}$$

Main Results

Definition of LDP

Definition

(Rate function) A function $I : \mathscr{E} \to [0, \infty)$ is called a rate function if it is lower semicontinuous. Moreover, I is a good rate function if for each constant $M < \infty$, the level set $\{x \in \mathscr{E} : I(x) \leq M\}$ is a compact subset of $\mathscr{E}$.

Definition

(Large deviation principle) Let I be a rate function on $\mathscr{E}$. Given a collection $\{\ell(\epsilon)\}_{\epsilon>0}$ of positive reals, a family $\{\mathbb{X}^\epsilon\}_{\epsilon>0}$ of $\mathscr{E}$-valued random variables is said to be satisfied a LDP on $\mathscr{E}$ with speed $\ell(\epsilon)$ and rate function I if the following two conditions hold:

(i) (Upper bound) For each closed subset $F \subset \mathscr{E}$,

$$\limsup_{\epsilon \to 0} \ell(\epsilon) \log \mathbb{P}(\mathbb{X}^\epsilon \in F) \leq -\inf_{x \in F} I(x).$$

(ii) (Lower bound) For each open subset $G \subset \mathscr{E}$,

$$\liminf_{\epsilon \to 0} \ell(\epsilon) \log \mathbb{P}(\mathbb{X}^\epsilon \in G) \geq -\inf_{x \in G} I(x).$$

The large deviation principle is equivalent to the following so-called Laplace principle.

Definition

(Laplace principle) Let I be a rate function on $\mathscr{E}$. Given a collection $\{\ell(\epsilon)\}_{\epsilon>0}$ of positive reals, a family $\{\mathbb{X}^\epsilon\}_{\epsilon>0}$ of $\mathscr{E}$-valued random variables is said to be satisfied the Laplace principle upper bound (respectively, lower bound) on $\mathscr{E}$ with speed $\ell(\epsilon)$ and rate function I if for all $\varrho \in C_b(\mathscr{E})$,

$$\limsup_{\epsilon\to 0} -\ell(\epsilon) \log \mathbb{E}\left[\exp\left(-\frac{\varrho(\mathbb{X}^\epsilon)}{\ell(\epsilon)}\right)\right] \leq \inf_{x\in\mathscr{E}} \{\varrho(x) + I(x)\}, \tag{17}$$

(respectively,

$$\liminf_{\epsilon\to 0} -\ell(\epsilon) \log \mathbb{E}\left[\exp\left(-\frac{\varrho(\mathbb{X}^\epsilon)}{\ell(\epsilon)}\right)\right] \geq \inf_{x\in\mathscr{E}} \{\varrho(x) + I(x)\}). \tag{18}$$

The Laplace principle is said to be held for $\{\mathbb{X}^\epsilon\}$ with speed $\ell(\epsilon)$ and rate function I if both the Laplace upper and lower bounds hold.

- A. Budhiraja and P. Dupuis, *Analysis and Approximation of Rare Events: Representations and Weak Convergence Methods*, Springer, 2019. [Theorems 1.5 and 1.8]
- P. Dupuis and R. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, John Wiley Sons, 2011. [Theorems 1.2.1 and 1.2.3]

For any $\epsilon > 0$, let $\mathcal{G}^\epsilon : C([0,T];\mathbb{R}^d) \to \mathscr{E}$ be a measurable map (with a slight abuse of notation $\mathcal{G}^\epsilon$). Next, we give the following sufficient condition for the Laplace principle (equivalently, the LDP) of $\mathbb{X}^\epsilon = \mathcal{G}^\epsilon(\epsilon^H B^H_\cdot)$ as $\epsilon \to 0$.

(A0) There exists a measurable map $\mathcal{G}^0 : I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d)) \to \mathscr{E}$ such that the following two conditions hold.

(i) Let $\{h^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ for any $M \in (0,\infty)$. If h^ϵ converges to h in distribution as S_M-valued random elements, then

$$\mathcal{G}^\epsilon\left(\epsilon^H B^H_\cdot + \epsilon^H/\ell^{\frac{1}{2}}(\epsilon)(R_H h^\epsilon)(\cdot)\right) \to \mathcal{G}^0(R_H h)$$

in law as $\epsilon \to 0$, where $\{\ell^\epsilon\}_{\epsilon>0}$ are positive reals.

(ii) For each $M \in (0,\infty)$, the set $\{\mathcal{G}^0(R_H h) : h \in S_M\}$ is a compact subset of $\mathscr{E}$.

Proposition

*If $\mathbb{X}^\epsilon_\cdot = \mathcal{G}^\epsilon(\epsilon^H B^H_\cdot)$ and **(A0)** holds, then the family $\{\mathbb{X}^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle (hence the LDP) on $\mathscr{E}$ with speed $\ell(\epsilon)$ and the rate function I given by*

$$I(f) = \inf_{\{h\in\mathcal{H}: f=\mathcal{G}^0(R_H h)\}} \left\{\frac{1}{2}\|h\|^2_{\mathcal{H}}\right\}, \ \ f \in \mathscr{E}. \quad (19)$$

Here we follow the convention that the infimum over an empty set is $+\infty$.

Below is a convenient and sufficient condition for verifying **(A0)**

(A1) There exists a measurable map $\mathcal{G}^0 : I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d)) \to \mathscr{E}$ for which the following two conditions hold.

(i) Let $\{h^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ for any $M \in (0,\infty)$. For each $\delta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\left(d\left(\mathcal{G}^\epsilon(\epsilon^H B^H_\cdot + \epsilon^H/\ell^{\frac{1}{2}}(\epsilon)(R_H h^\epsilon)(\cdot)), \mathcal{G}^0((R_H h^\epsilon)(\cdot))\right) > \delta\right) = 0,$$

where $d(\cdot,\cdot)$ stands for the metric on $\mathscr{E}$, $\{\ell^\epsilon\}_{\epsilon>0}$ are positive reals.

(ii) Let $\{h^n : n \in \mathbb{N}\} \subset S_M$ for any $M \in (0,\infty)$. If h^n converges to some element h in S_M as $n \to \infty$, then $\mathcal{G}^0(R_H h^n)$ converges to $\mathcal{G}^0(R_H h)$ in $\mathscr{E}$.

Proposition

If $\mathbb{X}^\epsilon_\cdot = \mathcal{G}^\epsilon(\epsilon^H B^H_\cdot)$ and **(A1)** *holds, then the family $\{\mathbb{X}^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle (hence the LDP) on $\mathscr{E}$ with speed $\ell(\epsilon)$ and the rate function I given by* (19).

Large deviation principle (LDP)

Introduce the following skeleton equation

$$\Upsilon_t^h = x + \int_0^t b(s, \Upsilon_s^h, \mathscr{L}_{X_s^0})\mathrm{d}s + \int_0^t \sigma(s, \mathscr{L}_{X_s^0})\mathrm{d}(R_H h)(s), \quad t \in [0, T], \tag{20}$$

where $h \in \mathcal{H}$ and X^0 is given

$$X_t^0 = x + \int_0^t b(s, X_s^0, \mathscr{L}_{X_s^0})\mathrm{d}s, \quad t \in [0, T].$$

As a consequence, we can define a map as follows

$$\mathcal{G}^0 : I_{0+}^{H+1/2}(L^2([0, T], \mathbb{R}^d)) \ni R_H h \mapsto \Upsilon^h \in C([0, T]; \mathbb{R}^d). \tag{21}$$

Our main result in this part reads as follows.

Theorem (LDP)

Assume that **(H1)** *holds. For each $\epsilon > 0$, let $X^\epsilon = \{X_t^\epsilon\}_{t\in[0,T]}$ be the solution to equation* (13). *Then the family $\{X^\epsilon : \epsilon > 0\}$ satisfies a LDP on $C([0, T]; \mathbb{R}^d)$ with speed ϵ^{2H} and the rate function I given by* (19)*, where $\mathcal{G}^0$ is defined in* (21)*.*

Moderate deviation principle (MDP)

In this part, we shall investigate the MDP for the equation (13) as $\epsilon \to 0$. The moderate deviations problem for $\{X^\epsilon : \epsilon > 0\}$ is to study the asymptotics of

$$\frac{1}{\kappa^2(\epsilon)} \log \mathbb{P}(Y^\epsilon \in \cdot),$$

where $\kappa(\epsilon) \to \infty, \epsilon^H \kappa(\epsilon) \to 0$ as $\epsilon \to 0$ and

$$Y^\epsilon := \frac{X^\epsilon - X^0}{\epsilon^H \kappa(\epsilon)}. \tag{22}$$

Recall the equations (13) and (16), from which we deduce that Y^ϵ satisfies

$$\begin{aligned} Y_t^\epsilon =& \frac{1}{\epsilon^H \kappa(\epsilon)} \int_0^t (b(t, X_s^0 + \epsilon^H \kappa(\epsilon) Y_s^\epsilon, \mathscr{L}_{X_s^\epsilon}) - b(s, X_s^0, \mathscr{L}_{X_s^0}))\mathrm{d}s \\ &+ \frac{1}{\kappa(\epsilon)} \int_0^t \sigma(s, \mathscr{L}_{X_s^\epsilon})\mathrm{d}B_s^H, \quad t \in [0, T]. \end{aligned} \tag{23}$$

Next, we put

$$\widetilde{\mathcal{G}}^{\epsilon}(\cdot) := \frac{\mathcal{G}^{\epsilon}(\cdot) - X^0}{\epsilon^H \kappa(\epsilon)},$$

which is a map from $C([0,T];\mathbb{R}^d)$ to $C([0,T];\mathbb{R}^d)$ such that $Y^{\epsilon} = \widetilde{\mathcal{G}}^{\epsilon}(\epsilon^H B^H_\cdot)$ due to the definition of $\mathcal{G}^{\epsilon}$ and the relation $X^{\epsilon}_\cdot = \mathcal{G}^{\epsilon}(\epsilon^H B^H_\cdot)$. Moreover, for any $h^{\epsilon} \in \mathcal{A}_M$, let

$$Y^{\epsilon,h^{\epsilon}}_\cdot = \widetilde{\mathcal{G}}^{\epsilon}\left(\epsilon^H B^H_\cdot + \epsilon^H \kappa(\epsilon)(R_H h^{\epsilon})(\cdot)\right), \tag{24}$$

then $Y^{\epsilon,h^{\epsilon}}$ solves the following equation

$$\begin{aligned} Y^{\epsilon,h^{\epsilon}}_t =& \frac{1}{\epsilon^H \kappa(\epsilon)} \int_0^t \left(b(s, X^0_s + \epsilon^H \kappa(\epsilon) Y^{\epsilon,h^{\epsilon}}_s, \mathscr{L}_{X^{\epsilon}_s}) - b(s, X^0_s, \mathscr{L}_{X^0_s}) \right) \mathrm{d}s \\ &+ \int_0^t \sigma(s, \mathscr{L}_{X^{\epsilon}_s}) \mathrm{d}(R_H h^{\epsilon})(s) + \frac{1}{\kappa(\epsilon)} \int_0^t \sigma(s, \mathscr{L}_{X^{\epsilon}_s}) \mathrm{d}B^H_s, \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s.. \end{aligned} \tag{25}$$

In additional to **(H1)**, we also need the following assumption.

(H2) The derivative $\nabla b(t,\cdot,\mu)(x)$ exists and there is a non-decreasing function $\widetilde{K}(t)$ such that for any $t \in [0,T], x, y \in \mathbb{R}^d, \mu \in \mathcal{P}_\theta(\mathbb{R}^d)$,

$$\|\nabla b(t,\cdot,\mu)(x) - \nabla b(t,\cdot,\mu)(y)\| \leq \widetilde{K}(t)(|x-y|).$$

Now, for each $h \in \mathcal{H}$, we introduce the following equation

$$\Xi_t^h = \int_0^t \nabla_{\Xi_s^h} b(s,\cdot,\mathscr{L}_{X_s^0})(X_s^0)\mathrm{d}s + \int_0^t \sigma(s,\mathscr{L}_{X_s^0})\mathrm{d}(R_H h)(s), \quad t \in [0,T], \tag{26}$$

which is used to give the rate function of Theorem 2.4 below. Under the time Hölder continuity of σ with order belonging to $(1-H,1]$, the equation (26) admits a unique solution. Therefore, this allows us to define a map as follows

$$\widetilde{\mathcal{G}}^0 : I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d)) \ni R_H h \mapsto \Xi^h \in C([0,T];\mathbb{R}^d). \tag{27}$$

Theorem (MDP)

Assume that **(H1)** *and* **(H2)** *hold. For each $\epsilon > 0$, let $Y^\epsilon = \{Y_t^\epsilon\}_{t\in[0,T]}$ be defined in* (22). *Then the family $\{Y^\epsilon : \epsilon > 0\}$ satisfies a LDP on $C([0,T];\mathbb{R}^d)$ with speed $\kappa^{-2}(\epsilon)$ and the rate function I given by*

$$I(f) = \inf_{\{h\in\mathcal{H}: f=\widetilde{\mathcal{G}}^0(R_H h)\}} \left\{\frac{1}{2}\|h\|_{\mathcal{H}}^2\right\}, \quad f \in C([0,T];\mathbb{R}^d), \tag{28}$$

where $\widetilde{\mathcal{G}}^0$ is defined in (27).

Central limit theorem (CLT)

This part is devoted to studying the CLT for equation (13). More precisely, we shall show that $\frac{X^\epsilon - X^0}{\epsilon^H}$ converges to a stochastic process in the pth-moment sense as $\epsilon \to 0$. The limit process is a solution to some linear equation which involves the Lions derivative of the coefficient b.
we will impose the following conditions on b and σ.

(H3) For every $t \in [0,T]$, $b(t,\cdot,\cdot) \in C^{1,(1,0)}(\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$, and there exists a non-decreasing function $\bar{K}(t)$ such that

(i) for any $t \in [0,T],\ x,y \in \mathbb{R}^d,\ \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\|\nabla b(t,\cdot,\mu)(x)\| + |D^L b(t,x,\cdot)(\mu)(y)| \le \bar{K}(t), \quad \|\sigma(t,\mu) - \sigma(t,\nu)\| \le \bar{K}(t)\mathbb{W}_\theta(\mu,\nu),$$

and $|b(t,0,\delta_0)| + \|\sigma(t,\delta_0)\| \le \bar{K}(t)$.

(ii) for any $t \in [0,T],\ x,y,z_1,z_2 \in \mathbb{R}^d,\ \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}&\|\nabla b(t,\cdot,\mu)(x) - \nabla b(t,\cdot,\nu)(y)\| + |D^L b(t,x,\cdot)(\mu)(z_1) - D^L b(t,y,\cdot)(\nu)(z_2)| \\ &\le \bar{K}(t)(|x-y| + |z_1 - z_2| + \mathbb{W}_\theta(\mu,\nu)).\end{aligned}$$

Our main result in this part is stated in the following theorem.

Theorem

Assume that **(H3)** *holds, then for any* $p \geq \theta$ *and* $p > 1/H$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\frac{X_t^\epsilon - X_t^0}{\epsilon^H} - Z_t\right|^p\right) \leq C_{T,p,H}\epsilon^{pH}\left(1 + \sup_{t\in[0,T]}|X_t^0|^{2p}\right), \quad \epsilon \in (0, \epsilon_0],$$

where Z_t *satisfies*

$$Z_t = \int_0^t \nabla_{Z_s} b(s,\cdot,\mathscr{L}_{X_s^0})(X_s^0)\mathrm{d}s + \int_0^t \left(\mathbb{E}\langle D^L b(s,u,\cdot)(\mathscr{L}_{X_s^0})(X_s^0), Z_s\rangle\right)|_{u=X_s^0}\mathrm{d}s$$
$$+ \int_0^t \sigma(s,\mathscr{L}_{X_s^0})\mathrm{d}B_s^H, \quad t \in [0,T], \qquad (29)$$

and $\epsilon_0 > 0$ *is a constant which will appear in the proof.*

Proof of LDP

it is enough to check that **(A1)** holds with $\mathcal{G}^\epsilon, \mathcal{G}^0$ and $\ell(\epsilon)$ given by (14), (21) and ϵ^{2H}.

Lemma

Suppose that σ *satisfies* **(H1)** *and* $\mu \in C([0,T];\mathscr{P}_p(\mathbb{R}^d))$ *with* $p \geq \theta$ *and* $p > 1/H$. *Then there is a constant* $C_{T,p,H} > 0$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t \sigma(s,\mu_s)\mathrm{d}B_s^H\right|^p\right) \leq C_{T,p,H}\int_0^T \|\sigma(s,\mu_s)\|^p \mathrm{d}s. \qquad (30)$$

The following lemma characterizes the difference between X^ϵ and X^0.

Lemma

Suppose that **(H1)** *holds. Then for any* $p \geq \theta$ *and* $p > 1/H$, *there exists a constant* $\epsilon_0 > 0$ *such that for every* $\epsilon \in (0, \epsilon_0]$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|X_t^\epsilon - X_t^0\right|^p\right) \leq C_{T,p,H}\epsilon^{pH}\left(1 + \sup_{t\in[0,T]}|X_t^0|^p\right).$$

$\mathrm{d}X_t^\epsilon = b(t,X_t^\epsilon,\mathscr{L}_{X_t^\epsilon})\mathrm{d}t + \epsilon^H\sigma(t,\mathscr{L}_{X_t^\epsilon})\mathrm{d}B_t^H,\ X_0^\epsilon = x.\quad X_t^0 = x + \int_0^t b(s,X_s^0,\mathscr{L}_{X_s^0})\mathrm{d}s,\ t\in[0,T].$

Proposition (To verify (A1) (i))

Suppose that **(H1)** *holds and let* $\{h^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ *for any* $M \in (0, \infty)$*. Then, for any* $\delta > 0$*,*

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\|X_\cdot^{\epsilon,h^\epsilon} - \mathcal{G}^0((R_H h^\epsilon)(\cdot))\|_\infty > \delta \right) = 0,$$

where $\|\cdot\|_\infty$ *is the uniform norm on* $C([0,T];\mathbb{R}^d)$.

Proof. For each fixed $\epsilon > 0$, we have

$$\begin{aligned} & X_t^{\epsilon,h^\epsilon} - \mathcal{G}^0((R_H h^\epsilon)(\cdot))(t) = X_t^{\epsilon,h^\epsilon} - \Upsilon_t^{h^\epsilon} \\ = & \int_0^t \left(b(s, X_s^{\epsilon,h^\epsilon}, \mathscr{L}_{X_s^\epsilon}) - b(s, \Upsilon_s^{h^\epsilon}, \mathscr{L}_{X_s^0}) \right) \mathrm{d}s \\ & + \int_0^t \left(\sigma(s, \mathscr{L}_{X_s^\epsilon}) - \sigma(s, \mathscr{L}_{X_s^0}) \right) \mathrm{d}(R_H h^\epsilon)(s) + \epsilon^H \int_0^t \sigma(s, \mathscr{L}_{X_s^\epsilon}) \mathrm{d}B_s^H, \quad t \in [0,T]. \end{aligned}$$

Then, it follows that

$$\begin{aligned} |X_t^{\epsilon,h^\epsilon} - \Upsilon_t^{h^\epsilon}|^2 \le & 3 \left| \int_0^t \left(b(s, X_s^{\epsilon,h^\epsilon}, \mathscr{L}_{X_s^\epsilon}) - b(s, \Upsilon_s^{h^\epsilon}, \mathscr{L}_{X_s^0}) \right) \mathrm{d}s \right|^2 \\ & + 3 \left| \int_0^t \left(\sigma(s, \mathscr{L}_{X_s^\epsilon}) - \sigma(s, \mathscr{L}_{X_s^0}) \right) \mathrm{d}(R_H h^\epsilon)(s) \right|^2 \\ & + 3\epsilon^{2H} \left| \int_0^t \sigma(s, \mathscr{L}_{X_s^\epsilon}) \mathrm{d}B_s^H \right|^2 =: J_1(t) + J_2(t) + J_3(t). \end{aligned} \tag{31}$$

Lemma

Suppose that **(H1)** *holds. Then for any $M > 0$,*

$$\sup_{h \in S_M} \sup_{t \in [0,T]} |\Upsilon_t^h|^2 \leq C_{T,H,M},$$

where $C_{T,H,M}$ is a positive constant only depending on T, H, M.

Proof.

$$\begin{aligned} |\Upsilon_t^h|^2 &= |x|^2 + 2\int_0^t \langle \Upsilon_s^h, b(s, \Upsilon_s^h, \mathscr{L}_{X_s^0}) \rangle \mathrm{d}s + 2\int_0^t \langle \Upsilon_s^h, \sigma(s, \mathscr{L}_{X_s^0}) \mathrm{d}(R_H h)(s) \rangle \\ &=: |x|^2 + I_1(t) + I_2(t). \end{aligned} \quad (32)$$

Proposition (To verify (A1)(ii))

Suppose that **(H1)** *holds and let* $\{h^n : n \in \mathbb{N}\} \subset \mathcal{S}_M$ *for any* $M \in (0, \infty)$ *such that* h^n *converges to element* h *in* S_M *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\mathcal{G}^0(R_H h^n)(t) - \mathcal{G}^0(R_H h)(t)| = 0.$$

Proof. For each $n \geq 1$, let Υ^{h^n} be the solution of equation (20) with h replaced by h^n. By (21), there hold $\mathcal{G}^0(R_H h^n) = \Upsilon^{h^n}$ and $\mathcal{G}^0(R_H h) = \Upsilon^h$.

1. We first prove that $\{\Upsilon^{h^n}\}_{n\geq 1}$ is relatively compact in $C([0,T];\mathbb{R}^d)$. With the help of the Arzelà-Ascoli theorem, it is enough to show that $\{\Upsilon^{h^n}\}_{n\geq 1}$ is uniformly bounded and equi-continuous in $C([0,T];\mathbb{R}^d)$.

a) By previous Lemma, there exists a constant $C_{T,H,M} > 0$ such that

$$\sup_{n\geq 1} \sup_{t\in[0,T]} |\Upsilon_t^{h^n}| \leq C_{T,H,M}, \tag{33}$$

which means that $\{\Upsilon^{h^n}\}_{n\geq 1}$ is uniformly bounded in $C([0,T];\mathbb{R}^d)$.

b) Equi-continuous of $\{\Upsilon^{h^n}\}_{n\geq 1}$ in $C([0,T];\mathbb{R}^d)$. By (20), we deduce that for $0 \leq s < t \leq T$,

$$\Upsilon_t^{h^n} - \Upsilon_s^{h^n} = \int_s^t b(r, \Upsilon_r^{h^n}, \mathscr{L}_{X_r^0})\mathrm{d}r + \int_s^t \sigma(r, \mathscr{L}_{X_r^0})\mathrm{d}(R_H h^n)(r). \tag{34}$$

2. Since $\{\Upsilon^{h^n}\}_{n\geq 1}$ is relatively compact in $C([0,T];\mathbb{R}^d)$, any subsequence of $\{\Upsilon^{h^n}\}_{n\geq 1}$, we can extract a further subsequence (not relabelled) such that Υ^{h^n} converges to some $\bar{\Upsilon}$ in $C([0,T];\mathbb{R}^d)$.
3. To show that $\bar{\Upsilon} = \Upsilon^h$. Then we can conclude that the full sequence Υ^{h^n} converges to Υ^h in $C([0,T];\mathbb{R}^d)$, which is the desired assertion.

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Thank you very much for your kind attention!